Signal Analysis & Communication ECE 355

Ch. 1.6: Properties of Systems (Contd.)

3. Causality
4. Stability
5. Time Invariance
6. Linearity

Lecture 6

20-09-2023

Ch. 1·6 PROPERTIES OF SYSTEMS (Contd.)

③ Causality

Defn: A sys. is causal if for all time $t \in \mathbb{R} / n \in \mathbb{Z}$, the output value $y(t) / y[n]$ depends only on the past & present input values $\{x(\tau)\}_{\tau \leq t} \,/\, \{x[n_0]\}_{n_0 \leq n}$.

- Meaning → the sys. doesnot look into future.
- You can often check causality by just inspecting the formula for $y(t)$.
- Examples: ① All memoryless systems are causal
  ② the Capacitor $v(t) = \int_{-\infty}^{t} \frac{1}{C} i(\tau) d\tau$ Causal
  ③ $y(t) = x(t+1)$ Non-Causal
  ④ $y[n] = \frac{1}{2m+1} \sum_{k=-m}^{m} x[n+k]$ Non-causal (indep. variable - spatial coordinates (not time))
  Moving Average - used in dig. image processing to smooth-out the fluctuations.

④ Stability

Defn.: A system is Bounded-I/p-Bounded-O/p stable if bounded I/p sig. i.e., $|x(t)| < \infty, \forall t \,/\, |x[n]| < \infty, \forall n$, leads to bounded O/p sig., i.e., $|y(t)| < \infty, \forall t \,/\, |y[n]| < \infty, \forall n$.

- Meaning → small I/Ps lead to O/Ps that do not diverge.
- Stability is a critical property in many engg. applications.
- Stability of phyical sys. results from the presence of mechanisms that dissipates energy.
- Examples: ① RLC Sys. (R dissipates energy) Stable
  ② Spring-mass Sys. Stable
  ③ $y[n] = \sum_{k=-\infty}^{n} x[k]$ Not stable (if $x[k] = 1, \forall k$) [the defn should be valid for all I/Ps]
  ④ $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$ Not stable (for eg., if $x(t) = u(t)$)

⑤ Time Invariance

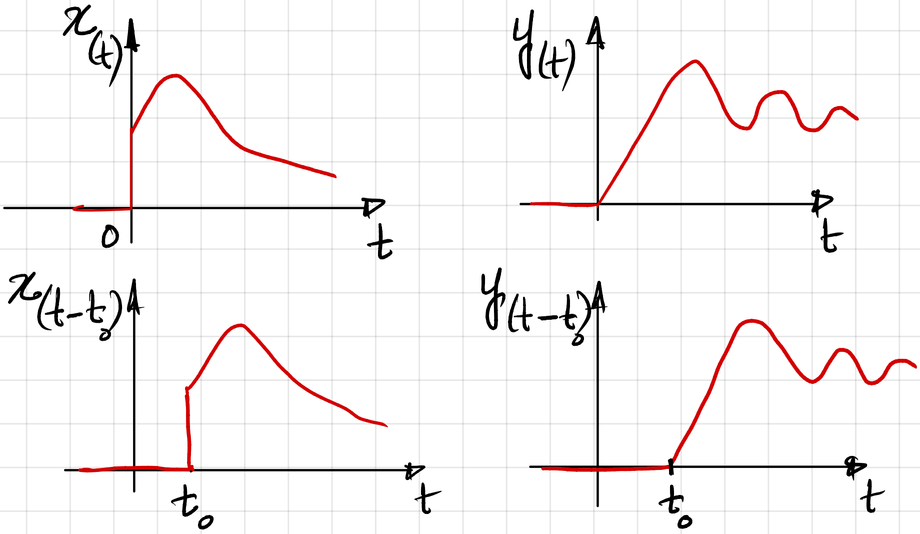
- A system is time invariant if a finite shift in I/P sig. results in an identical shift in the O/P sig.
- If

$$x(t) \,|\, x[n] \xrightarrow{S} y(t) \,|\, y[n]$$

then

$$x(t-t_0) \,|\, x[n-n_0] \longrightarrow y(t-t_0) \,|\, y[n-n_0]$$

for all $t, t_0 \in \mathbb{R}$, $n, n_0 \in \mathbb{Z}$



- Meaning → an experiment on the sys. tomorrow will produce the same results for an experiment on the system today.

A. $x(t) \longrightarrow$ [Delay] $\xrightarrow{x(t-t_0)}$ [S] $\longrightarrow y(t-t_0)$

$\updownarrow$ = Time-invariant

B. $x(t) \longrightarrow$ [S] $\xrightarrow{y(t)}$ [Delay] $\longrightarrow y(t-t_0)$

- Examples: ① $y(t) = x(2t)$

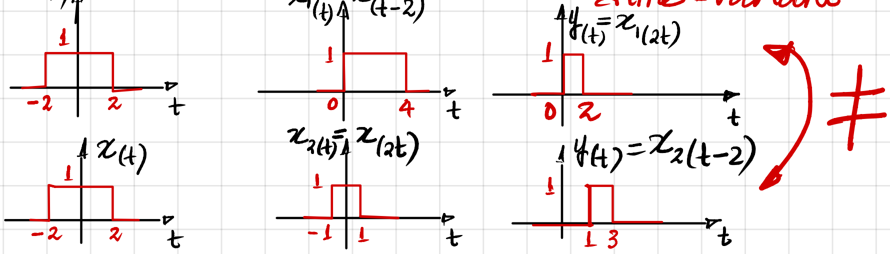
A. $x(t) \xrightarrow{\text{Delay}} x(t-t_0) \xrightarrow{S} x(\underbrace{2t}_{\text{Sys. Response}} - t_0)$

B. $x(t) \xrightarrow{S} x(2t) \xrightarrow{\text{Delay}} x(2(t-t_0))$

$\neq$ Time-variant

Delay.

Time-variant



② $y(t) = t\,x(t)$

A. $x(t) \xrightarrow{\text{Delay}} x(t-t_0) \xrightarrow{S} t\,x(t-t_0)$

B. $x(t) \xrightarrow{S} t\,x(t) \xrightarrow{\text{Delay}} (t-t_0)\,x(t-t_0)$

$\neq$

Time-variant (TV)

③ $y(t) = \sin(x(t))$ Time-invariant (TIV)

④ $y[n] = (x[n])^2$ Time-invariant

⑤ For RLC Circuits – Differential Eqns.

I. If coefficients of terms-constant $\Rightarrow$ TIV

II. If coefficients of terms-time varying. $\Rightarrow$ TV

⑥ Linearity

Defn.: A system is linear if the following conditions hold:

- For an arbitrary I/Ps $x_1(t) / x_1[n]$ & $x_2(t) / x_2[n]$

$$x_1(t) / x_1[n] \xrightarrow{S} y_1(t) / y_1[n]$$

$$x_2(t) / x_2[n] \xrightarrow{S} y_2(t) / y_2[n]$$

(i) $x_1(t) / x_1[n] + x_2(t) / x_2[n] \xrightarrow{S} y_1(t) / y_1[n] + y_2(t) / y_2[n]$

"Additivity"

(ii) $a x_1(t) / a x_1[n] \xrightarrow{S} a y_1(t) / a y_1[n]$

where $a \in \mathbb{C}$

"Homogeneity / Scaling"

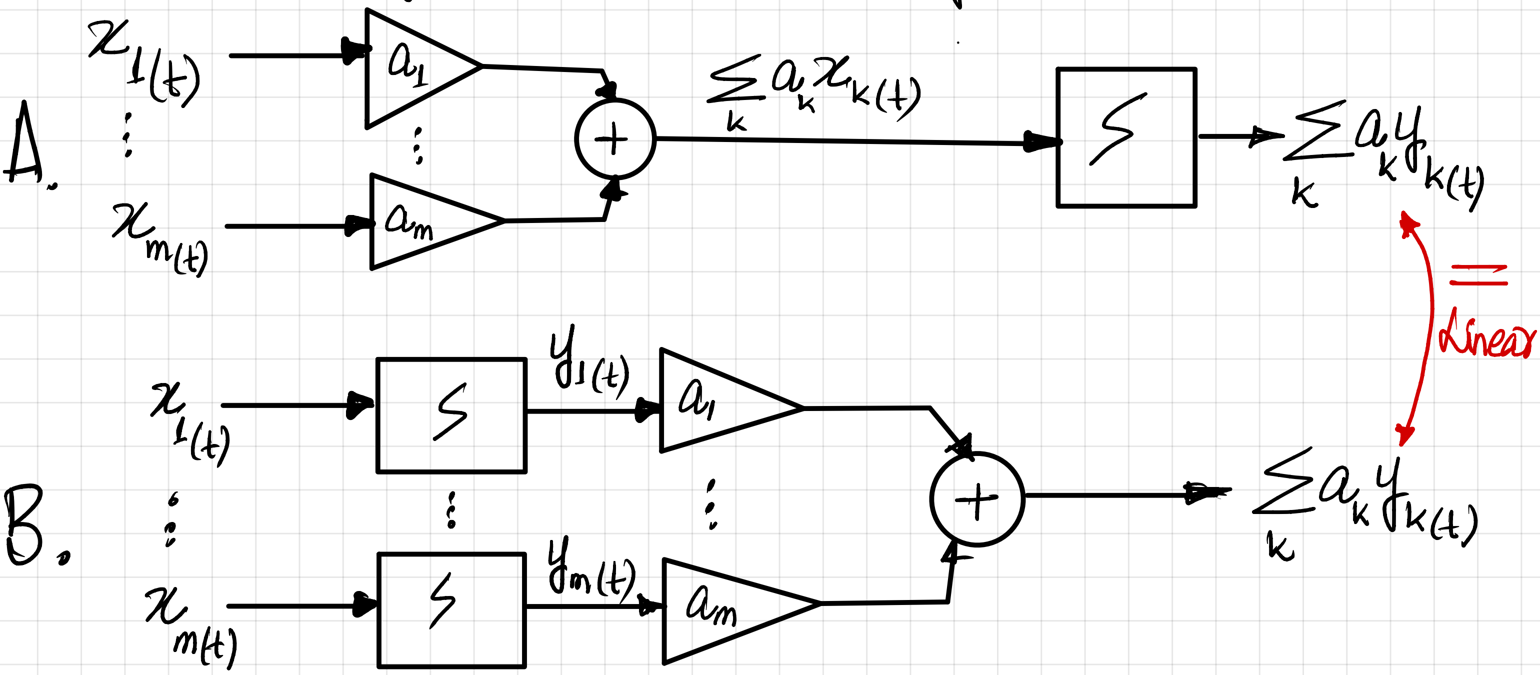
NOTE: A compact way to write this

$$a x_1(t) + b x_2(t) \xrightarrow{S} a y_1(t) + b y_2(t) \quad \forall a, b \in \mathbb{C}$$

$$a x_{1[n]} + b x_{2[n]} \xrightarrow{S} a y_{1[n]} + b y_{2[n]} \quad \forall a, b \in \mathbb{C}$$

OR $$\sum_k a_k x_{k(t)} \xrightarrow{S} \sum_k a_k y_{k(t)} \;\Big/\; \sum_k a_k x_{k[n]} \xrightarrow{S} \sum_k a_k y_{k[n]}$$
$$\forall a_k \in \mathbb{C}$$

* A Linear Sys. commutes with scaling & summation.



- Examples: ① $y_{(t)} = \sin(t)\, x_{(t)}$

A. $\rightarrow a x_{1(t)} + b x_{2(t)} \longrightarrow \sin(t)\left[a x_{1(t)} + b x_{2(t)}\right]$ $=$

B. $\rightarrow \sin(t)\, x_{1(t)}\, a + \sin(t)\, x_{2(t)}\, b \longrightarrow \sin_{(t)}\left[a x_{1(t)} + b x_{2(t)}\right]$

Linear

② $y_{[n]} = n x_{[n]}$

A. $\rightarrow a x_{1[n]} + b x_{2[n]} \longrightarrow n\left(a x_{1[n]} + b x_{2[n]}\right)$ $=$

B. $\rightarrow n x_{1[n]}\, a + n x_{2[n]}\, b \longrightarrow n\left(a x_{1[n]} + b x_{2[n]}\right)$

Linear

③ $y_{(t)} = x^2_{(t)}$

A. $\rightarrow a x_{1(t)} + b x_{2(t)} \longrightarrow \left(a x_{1(t)} + b x_{2(t)}\right)^2$ $\neq$

B. $\rightarrow x^2_{1(t)} \cdot a + x^2_{2(t)} \cdot b \longrightarrow a x^2_{1(t)} + b x^2_{2(t)}$

Nonlinear

④ $y_{(t)} = 2x_{(t)} + \underline{\underline{3}}$ Nonlinear

NOTE: For a linear system, we always have

$$x_{(t)} = 0 \xrightarrow{S} y_{(t)} = 0 \quad \forall t$$

- In linear Sys., zero I/p always produces zero O/P
- In example ④ above

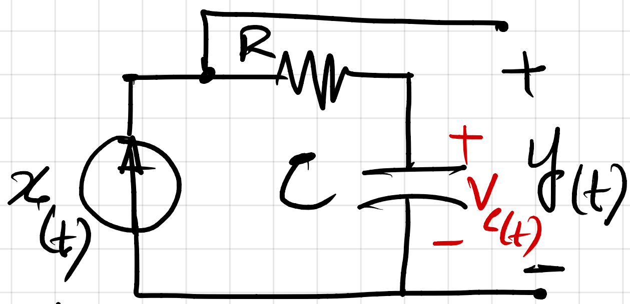
$$y_{(t)} = 2x_{(t)} + 3$$

- zero I/p here is not producing zero O/p here!
- Sys. is not linear.

$$A. \to x_{1(t)} + x_{2(t)} \longrightarrow 2(x_{1(t)} + x_{2(t)}) + \underline{\underline{3}} \quad \Big\} \neq$$

$$B \to (2x_{1(t)} + 3) + (2x_{2(t)} + 3) \longrightarrow 2(x_{1(t)} + x_{2(t)}) + \underline{\underline{6}}$$

- In RC Circuit



$$\int_0^t dy_{(\tau)} = Rx_{(t)} + \frac{1}{C}\int_0^t x_{(\tau)}\,d\tau$$

$$y_{(t)} = Rx_{(t)} + \underbrace{V_{c(0)}}_{} + \frac{1}{C}\int_0^t x_{(\tau)}\,d\tau$$

If it is zero $\to$ Sys. is linear

O/w $\to$ Sys. is Non-linear.

- Formally called 'initial rest condition'
- For any time t_0: if $x_{(t)} = 0$ for $t < t_0$ then $y_{(t)} = 0$ for $t < t_0$ — For a linear Sys.